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RAYMOND Y.T. WONG

PERIODIC ACTIONS ON (I-D) NORMED LINEAR SPACES

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PERIODIC ACTIONS ON (I-D) NORMED LINEAIR SPACES

Raymond Y.T. Wong *)

1. Introduction.

In this paper we study periodic homeomorphisms of a normed linear space (NLS) E which is homeomorphic ($\cong$) to F^{ω} or F^{ω}_f for some NLS F, where F^{ω} is the countable infinite product of F and F^{ω}_f is the subspace consisting of all points having at most finitely many non-zero coordinates. (It is known that the class of spaces E includes all separable infinite-dimensional (I-D) Fréchet spaces, all (I-D) Hilbert and reflexive Banach spaces,etc.) One of our main results (Theorem 1) states that any two fixed point free periodic homeomorphisms β, β_1 of prime period q on E (of E onto itself) are conjugate. We accomplish this by showing that their orbit spaces E/β, E/β_1 are homeomorphic (Corollary 1). In view of the classification theorems [8] and [9], we need only to show that they have the same homotopy type. Indeed we prove (Theorem 3) that each orbit space has the same homotopy type as the inductive limit, $\varinjlim S^{2n-1}/\alpha_n$, where S^{2n-1}/α_n is the orbit space of the period q homeomorphism α_n on the unit sphere $S^{2n-1} \subset C^n$ which takes each $(z_0,z_1,\dots)$ to $(e^{2\pi i/q}z_0, e^{2\pi i/q}z_1, \dots)$.

By basic covering theory, each orbit space E/β is an Eilenberg-MacLane space (see Spanier [14 - 424]) of type ($\mathbb{Z}q$, 1); that is, the fundamental group of E/β is isomorphic to $\mathbb{Z}q$, the integers modulo q and are trivial in all higher dimensions. Theorem 3 then applies to classify (Theorem 4) all E-manifolds which are Eilenberg-MacLane spaces of type ($\mathbb{Z}q$, 1), and in fact each such manifold can be represented as the orbit space of some fixed point free period q homeomorphism on E (Theorem 5). In view of a classification theorem for smooth l_2-manifolds ([6], [11]), some of our main results may be restated to obtain results in the category of C^{∞}-smooth l_2-manifolds (see for example, Theorem 2).

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V. Klee ([10]) has shown that any compact set in an (I-D) Hilbert space H may be the fixed points set of some periodic homeomorphism of H of any period. This result was generalized by West ([15]) to include all closed subsets of the space E. Non-trivial examples of periodic homeomorphisms of E are provided by the fact that E is homeomorphic ($\cong$) to $E \times [-1,1]^{\omega}$ together with the following theorem of West ([16]):
The product of a countably infinite collection of (non-degenerate) compact, contractible polyhedra is homeomorphic with $[-1,1]^{\omega}$.

We remark that our technique do not apply to dealing with periodic homeomorphisms which are not fixed point free. The situation is not known even for involutions (period 2 homeomorphisms) having exactly one fixed point (in [18] some partial results are obtained in this direction). Apparently one needs an appropriate version of non(almost)-manifold classification.

2. Statement of principle results

Hypothesis

(1) _Throughout this paper, let_ E _stand for a normed linear space_ (NLS) _which is homeomorphic to_ F^{ω} _or_ F_f^{ω} _for some_ NLS F.

(2) _Let_ $q > 1$ _stand for an arbitrary prime number._

For any space X, two homeomorphisms $f, g\colon X \to X$ are said to be _conjugate_ (or _equivalent_) provided there is a third homeomorphism $h\colon X \to X$ such that $h \circ f = g \circ h$.

Theorem 1 (_Conjugation_) _Any two fixed point free periodic homeomorphisms on_ E _of period_ q _are conjugate._

There is associated with a periodic homeomorphism β on a space X the orbit space X/β. If β is fixed point free, then the natural projection $p\colon X \to X/\beta$ is a q-fold covering map. If $X = E$, then E/β is a connected metrizable E-manifold whose fundamental group is isomorphic to $\mathbb{Z}_q$ (see for example, 2.7.6 and 2.7.8 of Spanier [14]) and are trivial in all higher dimensions. If β_1 is another fixed point free periodic homeomorphism on E of period q, we shall prove, in Corollary 1, that E/β and E/β_1 are homeomorphic. In fact, we show that there is a homeomorphism $h\colon E/\beta \to E/\beta_1$ which induces a fibre homeomorphism $h_*\colon E \to E$ satisfying $h_* \circ \beta = \beta_1 \circ h_*$. If we let $E = l_2$ (the separable Hilbert space of all square summable real sequences) and let β, β_1 be C^∞-smooth, then E/β and E/β_1 are C^∞-smooth l_2-manifolds for which the projections $p_1\colon E \to E/\beta$, $p\colon E \to E/\beta_1$ are also C^∞-smooth. In view of the classification for smooth l_2-manifolds ([6], [11]): *Every homotopy equivalence between C^∞-smooth l_2-manifolds is homotopic to a homeomorphism*, we can, with exactly the same argument, assume $h\colon E/\beta \to E/\beta_1$, as obtained above, is C^∞-smooth. Then h_* is necessarily C^∞-smooth since locally $h_* = p_1^{-1} \circ h \circ p$. Thus we have

Theorem 2. Let β, β_1 be fixed point free periodic C^∞-diffeomorphisms on l_2 of period q. Then there is a C^∞-diffeomorphism h_* on l_2 such that $h_* \circ \beta = \beta_1 \circ h_*$.

A real (or complex) Hilbert space H is the space $l_2(\mathcal{X})$ of all square summable real (respectively, complex) sequences indexed by an *infinite* abstract set $I(\mathcal{X})$ of cardinality $\mathcal{X}$. Denote $l_2 = l_2(\mathcal{X}_0)$. A point in $l_2(\mathcal{X})$ will be denoted by $(z_0, z_1, \ldots)$ where $i(k) \in I(\mathcal{X}_0)$. Since it is known that $H \cong H^\omega$([3]), Theorem 1 and 2 then apply to all spaces homeomorphic (diffeomorphic) with H (resp., l_2), in particular, the unit sphere S of H (Klee [10]. Indeed Bessaga has shown ([2]) that S is C^∞-diffeomorphic to H when $H = l_2$. These facts are useful since periodic homeomorphisms on S then induce to ones on H and there are several well-known canonical fixed point free periodic maps on S. In the following we consider, for H a complex Hilbert space, examples (A) for $q = 2$, the antipodal map $A : S \to S$ such that $A(z) = -z$; (B) let q_1, q_2, ... be positive integers relatively prime to q. Then for any $(z_0, z_1, \ldots) \in S$, define $A(z_0, z_1, \ldots) =$
$= (e^{2\pi i/q} z_0,\ e^{2\pi i/q_1} z_1,\ e^{2\pi i/q_2} z_2,\ \ldots)$; and (C) let $H = l_2$ (complex Hilbert space). Define $\alpha\colon S \to S$ by

$$\alpha(z_0, z_1, \ldots) = (e^{2\pi i/q} z_0,\ e^{2\pi i/q} z_1,\ \ldots).$$

The orbit space S/A in example (B) may be called an (I-D) version of generalized lens space and will be denoted by $L(q, q_1, q_2, \ldots)$. For $n \geq 1$, let C^n

denote the usual 2n-dimensional complex space in l_2 and S^{2n-1} its unit sphere (do not confuse the 1-sphere S^1 with S). Let $\alpha: S \to S$ be defined as in example (C). α, when restricted to S^{2n-1}, induces a period q homeomorphism $\alpha_n: S^{2n-1} \to S^{2n-1}$, n = 1, 2, ... Let $\varinjlim S^{2n-1}$ denote the inductive limit of $\{S^{2n-1}\}_{n\geq 1}$; that is, $\varinjlim S^{2n-1}$ is the CW complex which is the union of the sequence $S^1 \subset S^3 \subset \ldots$ topologized by the topology coherent with the collection $\{S^{2n-1}\}_{n\geq 1}$. Similarly there is an inductive limit, $\varinjlim S^{2n-1}/\alpha_n$, corresponding with the collection $\{S^{2n-1}/\alpha_n\}_{n\geq 1}$. By basic covering theory, the homotopy groups of $\varinjlim S^{2n-1}/\alpha_n$ are isomorphic to $\{\mathbb{Z}q,0,0,\ldots\}$ (hence is an Eilenberg-MacLane space of type $(\mathbb{Z}q, 1)$). In fact we prove

Theorem 3. Let M *be a metrizable connected* E-*manifold whose homotopy groups are isomorphic to* $\{\mathbb{Z}q,0,0,\ldots\}$. *Then* M *has the same homotopy type as the* CW *complex* $\varinjlim S^{2n-1}/\alpha_n$.

This together with the classification theorem of [8] and [9] then yields

Theorem 4 (*Classification*) *Let* M, M_1 *be metrizable connected* E-*manifolds each with homotopy groups isomorphic to* $\{\mathbb{Z}q,0,0,\ldots\}$. *Then* $M \cong M_1$.

Corollary 1. *Let* β, β_1: $E \to E$ *be fixed point free periodic homeomorphisms of period* q, *Then* $E/\beta \cong E/\beta_1$.

Let M be as in Theorem 3. The universal covering space $\tilde{M}$ of M is a homotopically trivial E-manifold such that the projection $p_1: \tilde{M} \to M$ is a q-fold covering map. Hence $\tilde{M} \cong E$ ([8]) and we have

Theorem 5 (*Representation*) *Let* M *be a metrizable connected* E-*manifold. Then there is a* q-*fold covering projection* p_1: $E \to M$ *and a fixed point free periodic homeomorphism* $\beta: E \to E$ *of period* q *such that* β *induces a homeomorphism* $\beta_*: E/\beta \to M$ *for which the following diagram commutes*

$$\begin{array}{ccc} E & \xrightarrow{\ \beta\ } & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_1} \\ E/\beta & \xrightarrow{\ \beta_*\ } & M \end{array}$$

3. Application and other results

For matter of convenience we introduce the category whose objects are pairs (X,β), (X_1,β_1), ... where X, X_1 are spaces equipped with periodic homeomorphisms β, β_1, ... of period q, and whose morphisms are maps $m: (X,\beta) \to (X_1,\beta_1)$ of pairs such that m is a map of X into X_1 which commutes with β, β_1; that is, $\beta_1 \circ m = m \circ \beta$. We can speak of m as an **imbedding** homeomorphism, etc.

For any map $h: X \to X$, denote by $fp(h)$ the set of fixed points of h. The reflection map $x \to -x$ of any topological vector space will always be denoted by γ.

Homeomorphism extension

Let X be a space homeomorphic to $X \times F$, F a TVS. We say a set $Y \subset X$ is F-deficient if there is a homeomorphism $h: X \to X\times F$ such that $h(Y) \subset X\times\{0\}$. (See [1] and [4] for the equivalence of F-deficiency with the concept of Z-sets of Anderson.)

Theorem 6. Let A be a closed H-deficient subset of a complex Hilbert space H. Then each period q homeomorphism β on A extends to a period q homeomorphism $\tilde{\beta}$ on H such that $fp(\beta) = fp(\tilde{\beta})$.

Proof. First we remark that for any metric locally convex TVS $F \cong F\times F$, by a technique of Klee ([10]), any homeomorphism between two closed F-deficient subsets of F extends to one on F. Denote by Δq the diagonal $\{(x,x,...): x \in K\}$ of $H^q = H\times H\times...\times H$ (q times).

Let $\phi: H \to H\times H$ be a homeomorphism such that $\phi(A) \subset H\times\{0\}$. For any $a \in A$, denote $\phi(a) = (a_0,0)$ and $\phi(\beta^n(a)) = (a_n,0)$, $n = 1, ..., q-1$. Define $m_1: A \to H^q\times H^q$ by $m_1(a) = (a_0,...,a_q) \times (0,0,...,0)$ and $\gamma_\Delta: H^q \to H^q$ by $\gamma_\Delta(z_0,z_1,...,z_{q-1}) = (z_{q-1},z_0,z_1,...,z_{q-2})$. Let $p_1: H^q \times H^q \to H^q$ be the projection onto the first factor. Denote $p_1 \circ m_1(fp(\beta))$ by K_1. Consider the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{m_1} & (H^q\backslash K)\times H^q \\ \downarrow \beta & & \downarrow \gamma_\Delta\times\gamma \\ A & \xrightarrow{m_1} & (H^q\backslash K)\times H^q \end{array}$$

where $K = \Delta_q \backslash K_1$ and γ the reflection on H^q.

It is elementary to note that Δ_q is a H-deficient subset of H^q. Thus K is a locally closed (that is, a difference of two closed sets, Δ_q and K_1) H-deficient subset of $H^q \simeq H$. Hence by Cutler ([5]), $H^q \backslash K \simeq H$. So let m_2: $(H^q \backslash K) \times H^q \to H$ be a homeomorphism. Using the remark above we can extend $m_2 \circ m_1$: $A \to H$ to a homeomorphism λ on H. Let $\beta_1 = m_2 \circ (\gamma_\Delta \times \gamma) \circ m_2^{-1}$: $H \to H$. It is clear that β_1 is a period q homeomorphism such that $fp(\beta_1) = m_2(K_1)$. Then $\tilde{\beta} = \gamma^{-1} \circ \beta_1 \circ \lambda$ is the required extension of β.

Closed imbeddings

Theorem 7. *Let X be a space which can be imbedded as a closed subset of a Hilbert space H. Then for any two fixed point free period q homeomorphisms β, β_1 on X, H respectively, there is a closed imbedding m: $(X,\beta) \to (H,\beta_1)$.*

Proof. Let m_1: $X \to H \times H$ be a closed imbedding such that $m(X) \subset \{0\} \times H$. By theorem 6 the induced map $m_1 \circ \beta \circ m_1^{-1}$: $m_1(X) \to m(X)$ extends to a fixed point free period q homeomorphism $\tilde{\beta}$ on $H \times H$. By theorem 1 there is a homeomorphism m_2: $(H \times H, \tilde{\beta}) \to (H, \beta)$. Then let $m = m_2 \circ m_1$.

Let X = M be a metrizable connected H-manifold. By Henderson-West [8 - Theorem 6] M can be imbedded as a closed sub-manifold of H. Hence the proof above also yields

Corollary 2. *Let β, β_1 be fixed point free period q homeomorphisms on M, H respectively. Then there is a closed imbedding m: $(M,\beta) \to (H,\beta_1)$ such that m(M) is a sub-manifold of H.*

Negligible subsets

Theorem 8. *Let K_1, K_2, ... be closed H-deficient subsets of H. Suppose β, β_1: $H \to H$ are fixed point free periodic homeomorphisms of period q such that $\beta(K) = K$, where $K = \bigcup_{i \geq 1} K_i$, then there is a homeomorphism m: $(H \backslash K, \beta) \to (H, \beta_1)$.*

Proof. By Cutler ([5 - Theorem 1]), there is a homeomorphism m_1: $H \backslash K \to H$. Let $\alpha_1 = m_1 \circ \beta|_{H \backslash K} \circ m_1^{-1}$. By Theorem 1 let m_2: $(H,\alpha_1) \to (H,\beta_1)$ be a homeomorphism. Then $m = m_2 \circ m_1$ is as required.

Homeomorphism spaces are contractible

For any space X, let $G_0(X)$ denote the subspace of G(X) consisting of all periodic homeomorphisms and $G_n(X) = \{\beta \in G_0(X)\colon \text{period } (\beta) = n,\ n \geq 1\}$.

Theorem 9. *For $k \geq 0$, each $G_k(E)$ is contractible and there is a contraction $\{\phi_t\}\colon G(E) \to G(E)$ such that $\{\phi_t|_{G_k(E)}\}$ induces a contraction for $G_k(E)$.*

Proof. The same as Theorem 6 of [18]. Renz in [13] shows that G(E) is contractible. If the construction of the contraction $\Phi(h,t)$ in [13 - 184] is replaced by

$$\Phi(h,t) = \phi^{(n)}(\cdot,t)^{-1} \circ h^{(n)} \circ \phi^{(n)}(\cdot,t),$$

then we get a contraction denoted by $\{\phi_t\}$ with the desired properties of the theorem.

Periodic stability of homeomorphisms

A subset K of a space X is said to be deformable if for each open set U in X, there is a $g \in G(X)$ such that $g(K) \subset U$. An open set $U \subset X$ is said to contain a dilation system if there is a sequence of pairwise disjoint open sets B_0, B_1, ... in U converging to a point $p \in U$ and a homeomorphism r supported in U such that $r(B_{i+1}) = B_i$, $i \geq 0$. We sometimes call $(B_i,r)_{i\geq 0}$ a dilation system in U.

Theorem 10. *Suppose X is a metric space in which every open set contains a dilation system. Let $N \subset G(X)$ be the normal subgroup consisting of all finite compositions of $g \in G(X)$ such that supp(g) is deformable. Then N is simple.*

Proof. The following proof is derived from a technique of Fisher (On the group of all homeomorphisms of a manifold, Trans. A.M.S., 97(1960), 193-212). Suppose $h(\neq \text{id}) \in G(X)$. Then for some open $U \subset X$, $h^{-1}(U) \cap U = \emptyset$. Let $(B_i,r)_{i\geq 0}$ be a dilation system in U. Denote $B = \bigcup_{i\geq 1} B_i$. Suppose $g \in G(X)$ such that $\text{supp}(g) \subset B_1$, then define $\emptyset\colon X \to X$ (supported in B) by $\emptyset|_{B_i} = r^{1-i} \circ g \circ r^{i-1}|_{B_i}$ for $i \geq 1$ ($r^0 = \text{id}$) and $\emptyset(x) = x$ otherwise. Note that $\emptyset|_{B_1} = g|_{B_1}$. Consider

$$w = (r^{-1}\circ\emptyset^{-1}\circ h^{-1}\circ\emptyset\circ r)(r^{-1}\circ h\circ r) \circ h^{-1} \circ (\emptyset^{-1}\circ h\circ\emptyset).$$

The same proof as [Fisher - p. 197] shows that $w = g$. If $g \in G(X)$ such that supp(g) is deformable, then by definition there is a $f \in G(X)$ such that $f(\text{supp}(g)) \subset B_1$. Thus $f \circ g \circ f^{-1}$ is supported in B_1 and $g = f^{-1} \circ (f\circ g\circ f^{-1}) \circ f$. It follows that each $g \in N$ is a finite composition of conjugations of $\{h,h^{-1}\}$. Now suppose N_0 is any normal subgroup of N containing an h other than the identity. By what we have just shown, each $g \in N$ is a member of N_0. Thus $N_0 = N$ and Theorem 10 is proved.

It is known that for X = Q, s or any normed linear space $E \simeq E^\omega$, G(X) is stable ([14], [22]), in the sense that every $f \in G(X)$ can be written as a finite composition $f_n \dots f_2 f_1$ of homeomorphisms of X such that each f_i is the identity on some non-void open subset of X. By well-known properties of X, it is routine to verify that

(1) if $f \in G(X)$ is the identity on some non-void open subset of X, then supp(f) is deformable.

(2) each open $U \subset X$ contains a dilation system.

Hence we have

Corollary 3. Let X be a space homeomorphic to Q, s or any normed linear space $E \simeq E^\omega$. Then G(X) is simple.

For each fixed $k \geq 0$, the collection of all finite compositions of members in $G_k(X)$ clearly forms a non-trivial normal subgroup of G(X). Hence $G_k(X)$ is entirely G(X), which proves

Theorem 11. Let X be as above. Then for any $h \in G(X)$ and any $k \geq 0$, there are $h_1, \dots, h_n \in G_k(X)$ such that $h = h_n \circ \dots \circ h_2 \circ h_1$.

4. The key lemma.

In this section a basic knowledge of covering theory is assumed. Two maps $f,g\colon X \to Y$ are homotopic relative $A \subset X$, written $f \sim g$ rel(A), if there is a homotopy $\{\lambda_t\}$ joining f and g such that $\lambda_t(a) = \lambda_0(a)$ for all $a \in A$, $t \in [0,1]$.

Let S be the unit sphere of the separable complex Hilbert space l_2 and let $\alpha: S \to S$, $\alpha_n: S^{2n-1} \to S^{2n-1}$ be defined as in section 2.

Lemma 1 (*The key lemma*) *Let* $p: E \to M$ *be a* q-*fold covering projection onto an* E-*manifold* M. *Then for any* $a_0 \in S^1$ (*the* 1-*sphere*) *and any two distinct points* $b_0, b_1 \in p^{-1}(b)$, $b \in M$, *there is a sequence of imbeddings* $f_n: S^{2n-1} \to M$, $n \geq 1$, *such that*

(1) $f_1(a_0) = b_0$, $f_1 \circ \alpha_1(a_0) = b_1$,

(2) for all $n \geq 1$, $f_{n+1}\big|_{S^{2n-1}} = f_n$ *and*

(3) $p \circ f_n(x) = p \circ f_n \circ \alpha_n(x)$ *for all* x,

$$\begin{array}{ccccc} S^{2n-1} & \xrightarrow{f_n} & E & & \\ & & & \searrow^{p} & \\ \downarrow{\scriptstyle \alpha_n} & & & & M \\ & & & \nearrow_{p} & \\ S^{2n-1} & \xrightarrow{f_n} & E & & \end{array}$$

To give a proof we need

Lemma 2. *Let* $M \subset E$ *be open and* (K,L) *be a finite simplicial pair. Suppose* $g: K \to M$ *is a map such that* $g|_L$ *is piecewise linear, then there is a piecewise linear* $\tilde{g}: K \to M$ *such that* $\tilde{g} \sim \tilde{g}$ rel(L) *and for* $x \neq y$, $\tilde{g}(x) = \tilde{g}(y)$ *only if* $\{x,y\} \in L$.

Proof. This is a routine consequence of the infinite dimensionality of M together with the linear structure on E.

Lemma 3. *Let* $p: E \to M$ *be as in Lemma* 1 *and* e_1, e_2 *be distinct points in* $p^{-1}(b)$ *for some* $b \in M$. *Suppose for* $i = 1, 2$, $\lambda_i: ([0,1],0) \to (E,e_i)$ *are liftings of an imbedding* $\lambda_0: ([0,1],0) \to (M,b)$, *then* $\lambda_1([0,1]) \cap \lambda_2([0,1]) = \emptyset$.

Proof. Suppose for some $x \neq 0$, $\lambda_1(x) = \lambda_2(x)$. Then the restrictions of $\{\lambda_i\}$ induce distinct maps $\tilde{\lambda}_i: ([0,x],x) \to (E,\lambda_i(x))$, which for both $i = 1$ and 2, is a lifting of $\lambda_0|_{[0,x]}: ([0,x], x) \to (M,\lambda_0(x))$. This is impossible.

Proof of Lemma 1. By Henderson-West ([8 - Theorem 7]) there is an open set $M_1 \subset E$ and a homeomorphism $h: M_1 \to M$. The usual technique of pull-back then induces a covering projection $p_1: E_1 \to M_1$ and a (fibre) homeomorphism $h_1: E_1 \to E$ satisfying $p \circ h_1 = h \circ p_1$. So, without loss of generality, we may suppose M is an open subset of E.

We shall construct the sequence $\{f_n\}$ by induction. First consider $n = 1$. Let $a_0 \in S_1$ be as given by the lemma. Denote $a_k = \alpha_1^k(a_0)$ (mod q), where α_1^k is the k-iterate $\alpha_1 \circ \alpha_1 \circ \dots \circ \alpha_1$ of α_1. Denote by $A[a_{n-1},a_n]$ the closed arc in S^1 joining a_{n-1} to a_n in the counter-clockwise direction. Let $\lambda: ([0,1],0) \to (E,b_0)$ be any map such that $\lambda(1) = b_1$. By Lemma 2 we may replace the loop $p \circ \lambda: ([0,1],0) \to (M,b)$ by a piecewise linear map $\lambda_0: ([0,1],0) \to (M,b)$ satisfying $\lambda_0 \sim p \circ \lambda$ rel(0,1) and for $x \neq y$, $\lambda_0(x) = \lambda_0(y)$ only if $\{x,y\} \subset \{0,1\}$. Denote by λ_0^q the usual composition $\lambda_0 * \lambda_0 * \dots * \lambda_0$ (q-times) in the homotopy group; that is, $\lambda_0^q: ([0,1],0) \to (M,b)$ is a map such that $\lambda_0^q(x) = \lambda_0(q(x-(k-1)/q))$ for $x = [(k-1)/q, k/q]$, $k = 1, 2, \dots, q$. Since $\pi_1(M) \approx \mathbb{Z}q$, $\lambda_0^q \sim c$ rel(0,1), where $c: [0,1] \to (M,b)$ is the constant map.

Let $\tilde{\lambda}: ([0,1],0) \to (E,b_0)$ be the lifting of λ_0^q. Then $\tilde{\lambda}(0) = \tilde{\lambda}(1)$ and we assert that for $x \neq y$, $\tilde{\lambda}(x) = \tilde{\lambda}(y)$ only if $\{x,y\} \subset \{0,1\}$.To see this suppose $0 \le x \le y \le 1$ and $\tilde{\lambda}(x) = \tilde{\lambda}(y)$. By definition of λ_0^q, there are points $x_0,y_0 \in [0,1]$ such that $\lambda_0(x_0) = \lambda_0^q(x)$ and $\lambda_0(y_0) = \lambda_0^q(y)$. So $\lambda_0(x_0) = \lambda_0^q(x) = p \circ \tilde{\lambda}(x) = p \circ \tilde{\lambda}(y) = \lambda_0^q(y) = \lambda_0(y_0)$. Hence $x_0,y_0 \in \{0,1\}$. This implies that x and y both belonged to the end-point sets of the interval in which they are contained. So for some $0 \le m < n \le q$, $x = m/q$ and $y = n/q$. Since $\tilde{\lambda}(m/q) = \tilde{\lambda}(n/q)$, $[\lambda_0^{n-m}] = [\lambda_0^q|_{[m/q,n/q]}] = [c]$, where $[\cdot] \in \pi_1(M)$ and $[c]$ is the class of constant map c. Thus $[\lambda_0^{q-(n-m)}] = [\lambda^{q-(n-m)} * c] = [\lambda_0^{q-(n-m)} * \lambda^{n-m}] = [c]$. Since q is a prime and $[\lambda_0] \neq [c]$, this is possible only if $q - (n-m) = 0$ or $x = 0$, $y = 1$. Hence $\{x,y\} \subset \{0,1\}$ and our assertion is verified.

Define $f_1: S^1 \to E$ as follows. Fixed a homeomorphism $\mu: A[a_0,a_1] \to [0,1/q]$ with $\mu(a_0) = 0$. For any $x \in A[a_{n-1},a_n]$, let $x_0 \in A[a_0,a_1]$ be the unique point such that $\alpha_1^{n-1}(x_0) = x$. Then let $f_1(x) = \tilde{\lambda}(\mu(x\ ) + (n-1)/q)$. It is routine to verify that f_1 is an imbedding which satisfies $f_1(a_0) = b_0$, $f_1(a_1) = b_1$ and $p \circ f_1(x) = p \circ f_1 \circ \alpha_1(x)$ for all $x \in S^1$ as required by the lemma.

Now suppose $f_k\colon S^{2k-1} \to E$ has been constructed. Since $p \circ f_1\colon S^1 \to M$ is piecewise linear, we may require that in addition $p \circ f_k$ is piecewise linear and we shall construct f_{k+1} for the lemma such that $p \circ f_{k+1}$ is also piecewise linear. Denote by C^k the product $C_1 \times C_2 \times \cdots \times C_k$ of complex spaces $C_i = C$. Recall that C^k is regarded as $C^k \times 0 \subset C^k \times C_{k+1}$ and C_{k+1} is regarded as $0 \times C_{k+1} \subset C^k \times C_{k+1} = C^{k+1}$. Let S_0 denote the unit 1-sphere of C_{k+1}. For any $z \in S_0$, let

$$L(z) = \{(sz_1,tz) \in S^{2k+1}\colon z_1 \in S^{2k-1},\ s,\ t \in [0,1] \text{ and } s^2 + t^2 = 1\}.$$

We may view $L(z)$ as a cone over S^{2k-1} with vector $\{z\}$. By the definition of $\alpha_{k+1}\colon S^{2k+1} \to S^{2k+1}$, $\alpha_{k+1}(S_0) = S_0$ and the action $\alpha_{k+1}|_{S_0}$ is the same as α_1 on S^1. Fix any $c_0 \in S_0$. Let $c_n = \alpha^n_{k+1}(c_0)$ (mod q) and let $A[c_{n-1},c_n]$ be defined the same way as $A[a_{n-1},a_n]$. Denote $L[c_{n-1},c_n] = \cup\{L(z)\colon z \in A[c_{n-1},c_n]\}$. Clearly $S^{2k+1} = \bigcup_{n=1}^{q} L[c_{n-1},c_n]$ and $\alpha^{n-1}_{k+1}\,(L[c_0,c_1]) = L[c_{n-1},c_n]$.

Using the existence of f_1 and the infinite-dimensionality of M, we can extend $f_k\colon S^{2k-1} \to E$ to an imbedding $f'_k\colon S^{2k-1} \cup S_0 \to E$ such that $p \circ f'_k$ is piecewise linear and satisfies $p \circ f'_k(x) = p \circ f'_k \circ \alpha_{k+1}(x)$ for all x. Let $d_n = f'_k(c_n)$ (mod q). Then for some $d \in M$, $\{d_n\} \subset p^{-1}(d)$. By the linear structure on E, we can extend $f'_k|_{S^{2k-1}\cup\{c_0\}}$ to a map $g_0\colon (L(c_0),c_0) \to (E,d_0)$.

By Lemma 2 we may replace the map $p \circ g_0\colon (L(c_0),c_0) \to (M,d)$ by a piecewise linear map $g_1\colon(L(c_0),c_0) \to (M,d)$ such that $g_1 \sim p \circ g_0 \text{ rel}(S^{2k-1} \cup \{c_0\})$ and for $x \neq y$ in $L(c_0)$, $g_1(x) = g_1(y)$ only if $x,y \in S^{2k-1}$. Since $L(c_0)$ is simply connected, we can lift g_1 to a map $\tilde{g}_1\colon (L(c_0),c_0) \to (E,d_0)$. We verify easily that $\tilde{g}_1$ is an extension of $f'_k|_{S^{2k-1}\cup\{c_0\}}$ and by hypothesis of g_1, $\tilde{g}_1$ is an imbedding.

Define $g_n\colon (L(c_n),c_n) \to (M,d)$ by $g_n(x) = g_1 \circ \alpha^{-n}_{k+1}(x)$. Let $\tilde{g}_n\colon(L(c_n),c_n) \to (E,d_n)$ denote the lifting of g_n. Then each $\tilde{g}_n$ is an imbedding extending $f'_k|_{S^{2k-1}\cup\{c_n\}}$ and satisfies $p \circ \tilde{g}_n(x) = p \circ \tilde{g}_{n+1} \circ \alpha_{k+1}(x)$ for all $x \in L(c_n)$. We assert that $\tilde{g}_n(x) = \tilde{g}_m(y)$ only if $x = y$. To see this suppose for some $x \in L(c_m)$, $y \in L(c_n)$, $\tilde{g}_m(x) = \tilde{g}_n(y)$. There are points $x_0,y_0 \in L(c_0)$ for which $\alpha^m_{k+1}(x_0) = x$, $\alpha^n_{k+1}(y_0) = y$. Thus

$g_1(x_0) = g_m(x) = p \circ \tilde{g}_m(x) = p \circ \tilde{g}_n(y) = g_n(y) = g_1(y_0)$. Then

(1) If $x_0 \neq y_0$, by hypothesis of g_1, $\{x_0, y_0\} \subset S^{2k-1}$. Hence $\{x,y\} \in S^{2k-1}$. but since $f'_k(x) = \tilde{g}_m(x) = \tilde{g}_n(y) = f'_k(y)$, we have $x = y$.

(2) If $x_0 = y_0$, we may suppose $x_0 \notin S^{2k-1} \cup S_0$ (otherwise the conclusion follows easily). Let L be an arc in $L(c_0)$ joining x_0 and c_0 such that $g_1|_L: (L,c_0) \to (M,d)$ is an imbedding. Then the restrictions $\tilde{g}_m \circ \alpha^m_{k+1}|_L: (L,c_0) \to (E,d_m)$ and $\tilde{g}_n \circ \alpha^n_{k+1}|_L: (L,c_0) \to (E,d_n)$ are both liftings of $g_1|_L$. Since we assume $\tilde{g}_m \circ \alpha^m_{k+1}(x_0) = \tilde{g}_m(x) = \tilde{g}_n(y) = \tilde{g}_n \circ \alpha^n_{k+1}(x_0)$, by Lemma 3 this is the case only when $m = n \pmod q$. But $\tilde{q}_n$ is one-to-one, so $x = y$.

Define $\tilde{f}_k: (\bigcup_{n=0}^{q-1} L(c_n)) \cup S_0 \to E$ by $\tilde{f}_k|_{L(c_n)} = \tilde{g}_n$ and $\tilde{f}_k|_{S_0} = \tilde{f}'_k|_{S_0}$. By what we have just shown, $\tilde{f}_k$ is an imbedding which satisfies $p \circ \tilde{f}_k(x) = p \circ \tilde{f}_k \circ \alpha_{k+1}(x)$ for all x.

We shall employ exactly the same process to obtain an extension $f_{k+1}: S^{2k+1} \to E$. Let $h_0: (L[c_0,c_1],c_0) \to (E,d_0)$ be any map extending $\tilde{f}_k|_{B_1}$, where $B_1 = L(c_0) \cup L(c_1) \cup A[c_0,c_1]$. By Lemma 2 we may replace the map $p \circ h_0: (L[c_0,c_1],c_0) \to (M,d)$ by a piecewise linear map $h_1: (L[c_0,c_1],c_0) \to (M,d)$ such that $h_1 \sim p \circ h_0$ rel(B_1) and for $x \neq y$ in $L[c_0,c_1]$, $h_1(x) = h_1(y)$ only if $x,y \in B_1$. For $n = 1, 2, \ldots, q-1$, define $h_n: (L[c_{n-1},c_n],c_{n-1}) \to (M,d)$ by $h_n(x) = h_1 \circ \alpha^{-(n-1)}_{k+1}(x)$ (α^0_{k+1} = identity). Since $L[c_{n-1},c_n]$ is simply connected, we can lift h_n to a map $\tilde{h}_n: (L[c_{n-1},c_n],c_{n-1}) \to (E,d_{n-1})$. We verify easily that each $\tilde{h}_n$ is an imbedding extending $\tilde{f}_k|_{B_n}$, where $B_n = L(c_{n-1}) \cup L(c_n) \cup A[c_{n-1},c_n]$, and satisfies $p \circ \tilde{h}_n(x) = p \circ \tilde{h}_{n+1} \circ \alpha_{k+1}(x)$ for all $x \in L[c_{n-1},c_n]$. It follows from exactly the same argument as for maps $\{\tilde{g}_n\}$ that $\{\tilde{h}_n\}$ satisfies $\tilde{h}_n(x) = \tilde{h}_m(y)$ only if $x = y$. Define $f_{k+1}: S^{2k+1} \to E$ by $f_{k+1}|L[c_{n-1},c_n] = \tilde{h}_n$. Then f_{k+1} extends f_k and fulfills all the requirements of the lemma.

5. Proof of Theorems 1, 3 and 5

Proof of Theorem 3. As pointed out in the discussion following the statement of Theorem 4, there is a q-fold covering projection $p: E \to M$. Fix

$a_0 \in S^1$ and distinct points $b_0, b_1 \in p^{-1}(b)$, $b \in M$. Let $f_n: S^{2n-1} \to M$ be a sequence of imbeddings satisfying conditions (1) - (3) of Lemma 1. By the usual technique of stereo-projection of $S \setminus \{point\}$ onto a hyperplane of H, we see that $(\varinjlim S^{2n-1}) \setminus \{point\} \simeq \varinjlim E^n$, where $E^1 \subset E^2 \subset \ldots$ are finite dimensional subspaces of H. Consequently $\varinjlim S^{2n-1}$ is homotopically trivial. By conditions (2) and (3) of Lemma 1, $\{f_n\}$ induces one-to-one maps $\tilde{f}: \varinjlim S^{2n-1} \to E$ and $f: \varinjlim S^{2n-1}/\alpha_n \to M$ satisfying $p \circ \tilde{f} = f \circ p_0$, where $p_0: \varinjlim S^{2n-1} \to \varinjlim S^{2n-1}/\alpha_n$ is the natural projection.

We claim that f is a weak homotopy equivalence. It suffices to show that $f_\#: \pi_1(\varinjlim S^{2n-1}/\alpha_n) \to \pi_1(M)$ is an isomorphism. To see this let $a = p_0(a_0)$. First we show $f_\#$ is one-to-one. Suppose $\lambda: ([0,1],0) \to (\varinjlim S^{2n-1}/\alpha_n, a)$ is a loop such that $f \circ \lambda \sim c$ rel(0,1) where $c: ([0,1],0) \to (M,b)$ is the constant map. Then λ and $f \circ \lambda$ lift to maps $\lambda_0: ([0,1],0) \to (\varinjlim S^{2n-1}, a_0)$ and $\lambda_1: ([0,1],0) \to (E,b_0)$ respectively such that $\lambda_1(1) = b_0$. But $\tilde{f} \circ \lambda_0$ is another lifting of $f \circ \lambda$, hence $\tilde{f} \circ \lambda_0 = \lambda_1$. Since $\tilde{f}|_{\lambda_0([0,1])}$ is an imbedding, $\lambda_0(1) = a_0$. We have shown that $\varinjlim S^{2n-1}$ is homotopically trivial, hence λ belongs to the homotopy class of the constant loop. Thus $f_\#$ is one-to-one. Next suppose $\mu: ([0,1],0) \to (M,b)$ is any loop. Denote the lifting of μ to (E,b_0) by $\tilde{\mu}$. Then $\tilde{\mu}(0) = b_0$ and $\mu(1) = b_i$ for some $b_i \in p^{-1}(b) \subset f_1(S^1)$. Let $\mu_0: ([0,1],0) \to (S^1, a_0)$ be any map for which $\mu_0(1) = f_1^{-1}(b_i) = \tilde{f}^{-1}(b_i)$. By the linear structure on E, $\tilde{f} \circ \mu_0 \sim \tilde{\mu}$ rel(0,1). This implies $f \circ p_0 \circ \mu_0 \sim \mu$. So $f_\#$ is onto and the claim is complete.

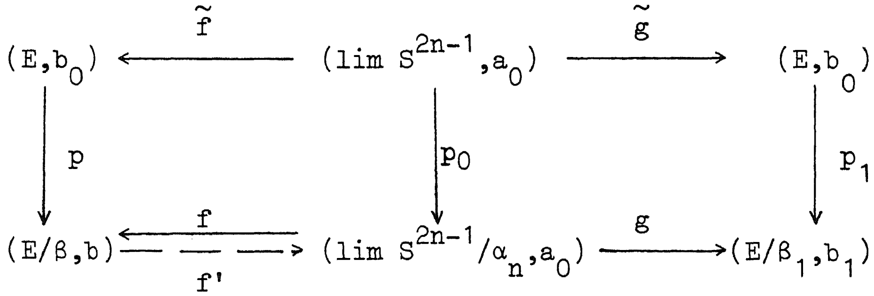
By Palais ([12 - Theorem 14]) and Whitehead ([17 - Theorem 1]), f is in fact a homotopy equivalence.

To prove Theorem 1 we need

Lemma 4. *Let X, X_1 be connected Hausdorff spaces carrying respectively fixed point free period q homeomorphisms β and β_1. Suppose $h: X \to X_1$ is an imbedding such that (i) for each $x \in X$, there is an $n \geq 1$ for which $h \circ \beta(x) = \beta_1^n \circ h(x)$ and (ii) there is a point $a_0 \in X$ such that $h \circ \beta(a_0) = \beta_1 \circ h(a_0)$, then $h \circ \beta(x) = \beta_1 \circ h(x)$ for all x.*

Proof. Let $A_n = \{x \in X: h \circ \beta(x) = \beta_1^n \circ h(x)\}$. Then each A_n is closed and $\{A_n\}$ are pairwise disjoint (mod q). Since X is connected, $X = A_n$ for some n. By hypothesis (ii), n = 1.

Proof of Theorem 1. Let $\beta, \beta_1: E \to E$ be fixed point free periodic homeomorphisms of period q. Fix any $a_0 \in S^1$ and $b_0 \in E$. By Lemma 1 there are one-to-one maps $\tilde{f}, \tilde{g}: (\varinjlim S^{2n-1}, a_0) \to (E, b_0)$ satisfying $\tilde{f} \circ \alpha_1(a_0) = \beta(b_0)$, $\tilde{g} \circ \alpha_1(a_0) = \beta_1(b_0)$ and such that $\tilde{f}$, $\tilde{g}$ induce homotopy equivalences $f: \varinjlim S^{2n-1}/\alpha_n \to E/\beta$ and $g: \varinjlim S^{2n-1}/\alpha_n \to E/\beta_1$. Let $a = p_0(a_0)$, $b = p(b_0)$ and $b_1 = p_1(b_0)$, where $p_0: \varinjlim S^{2n-1} \to \varinjlim S^{2n-1}/\alpha_n$, $p: E \to E/\beta$ and $p_1: E \to E/\beta_1$ are projections. Let $f': (E/\beta, b) \to \varinjlim S^{2n-1}/\alpha_n$ be a map such that $f' \circ f \sim$ identity and $f \circ f' \sim$ identity.



By [8], $g \circ f'$ is homotopic to a homeomorphism $h: (E/\beta, b) \to (E/\beta_1, b_1)$. Since E is simply connected, h induces a homeomorphism $h_*: (E, b_0) \to (E, b_0)$ such that $p_1 \circ h_* = h \circ p$. The usual construction of h_* goes as follows. Let any $x \in E$, let $\mu: ([0,1], 0) \to (E, b_0)$ be any map such that $\mu(1) = x$. The composition $h \circ p \circ \mu: ([0,1], 0) \to (E/\beta, b_1)$ lifts to a map $\tilde{\mu}: ([0,1], 0) \to (E, b_0)$. Define $h_*: E \to E$ by $h_*(x) = \tilde{\mu}(1)$. Then h_* has the required properties.

For any $x \in E$, there is an $i \geq 1$ for which $h_* \circ \beta(x) = \beta_1^i \circ h(x)$. We assert that i = 1 for all x (hence proving Theorem 1). In view of Lemma 4, we need only to verify $h_* \circ \beta(b_0) = \beta_1 \circ h_*(b_0)$. To see this suppose $\lambda: ([0,1], 0) \to (E, a_0)$ is a map such that $\lambda(1) = \alpha_1(a_0)$. Then $\mu = \tilde{f} \circ \lambda$ and $\mu_1 = \tilde{g} \circ \lambda$ are maps satisfying $\mu(1) = \beta(b_0)$ and $\mu_1(1) = \beta_1(b_0)$. Since $f' \circ p \circ \tilde{f} \circ \lambda \sim p_0 \circ \lambda$,

$$\begin{aligned} & h \circ p \circ \mu \\ & \sim g \circ f' \circ p \circ \tilde{f} \circ \lambda \\ & = g \circ f' \circ f \circ p_0 \circ \lambda \\ & \sim g \circ p_0 \circ \lambda \\ & = p_1 \circ \tilde{g} \circ \lambda \\ & = p_1 \circ \mu_1 . \end{aligned}$$

Hence

$h_* \circ \beta(b_0) = \mu_1(b_0) = \beta_1(b_0) = \beta_1 \circ h_* (b_0)$.

Proof of Theorem 5. Let p_1: $E \to M$ be given by the theorem. For any fixed point free period q homeomorphism β_0 on E (see West ([15]) for the existence of β_0), let p: $E \to E/\beta_0$ be the projection. By Theorem 4 there is a homeomorphism h: $E/\beta_0 \to M$. h then induces a fibre homeomorphism h_*: $E \to E$. Let $\beta = h_* \circ \beta_0 \circ h_*^{-1}$. β satisfies the requirements of the theorem.

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